泰勒級數

110-2 中原大學 微積分教學小組* 製

函數 f 在 x_0 的線性近似函數為 $f(x_o)+f'(x_0)(x-x_0)$。也就是，我們可用一次多項式 $L(x)=f(x_o)+f'(x_0)(x-x_0)$ 來估計在 x_0 附近的函數值 $f(x)$。一般而言，我們可用 n 次多項式更精確地來估計 $f(x)$；此多項式稱為泰勒多項式 (*Taylor polynomial*)。

Definition 1 If a function f can be differentiated n times at c, the nth *Taylor polynomial* for f at c is

$$P_n(x)=f(c)+f'(c)(x-a)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n. \tag{1}$$

- Note that $P_n(c)=f(c)$ and $P_n^{(k)}(c)=f^{(k)}(c)$ for $k=1,2,\cdots,n$.
- When $c=0$, the Taylor polynomial is also called the *Maclaurin polynomial*; that is, the nth *Maclaurin polynomial* for f is

$$P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n. \tag{2}$$

Example 1: Find the 6th Taylor polynomials at $\frac{\pi}{2}$ for (a) $\sin x$ (b) $\cos x$.

Solution: (a) $P_6(x)=1-\frac{1}{2}(x-\frac{\pi}{2})^2+\frac{1}{24}(x-\frac{\pi}{2})^4-\frac{1}{720}(x-\frac{\pi}{2})^6$,

(b) $P_6(x)=-(x-\frac{\pi}{2})+\frac{1}{6}(x-\frac{\pi}{2})^3-\frac{1}{120}(x-\frac{\pi}{2})^5$.

Example 2: Find the nth Taylor polynomial for $1/x$ at 1.

Solution: $P_n(x)=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n$.

Example 3: Find the 6th Maclaurin polynomials for $f(x)=$ (a) $\cos x$ (b) $\frac{1}{1-x}$ (c) e^x .

Solution: (a) $P_6(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$. (b) $P_6(x)=1+x+x^2+x^3+x^4+x^5+x^6$.

(c) $P_6(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}$.

Definition 2 If $a_0,a_1,a_2,\ldots$ are constants and x is a variable, then a series of the form

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1(x-c)+a_2(x-c)^2+\cdots+a_n(x-c)^n+\cdots$$

is called a *power series* (冪級數) in x.

Definition 3 If $f^{(n)}(c)$ exists for all $n=1,2,3,\cdots$, the *Taylor series*† (泰勒級數) of $f(x)$ at c is

$$f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots \tag{3}$$

- When $c=0$, the Taylor series is also called the *Maclaurin series*.

*本文件由應數系教師林余昭博士 (yujaulin@cycu.edu.tw) 主編，為理工微積分 3 學分課程之兩小時 統一教學教材。

†無窮 (項) 級數之收斂性檢定法請參閱教科書 (Calculus, Ron Larson, 11th ed. metric version) §9.1∼ §9.6。

此類級數屬於 power series 的一種，泰勒多項式 T_n 可視為泰勒級數的前 n 項的和。

冪級數的例子： $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$, $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ 。

Example 4: Express $\frac{1}{1+x^2}$ as a power series if $-1 < x < 1$.

Solution: Using the fact $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots .$$

Binomial Series (二項式級數) If k is a real number and $-1 < x < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots . \tag{4}$$

當 k 為一正整數，二項級數即為二項展開式 (binomial expansion)，即

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n \text{，} \quad \text{其中} \binom{k}{n} = \mathrm{C}_n^k = \frac{k!}{n!\,(k-n)!} \text{，且定義 } 0! = 1 \text{。} \tag{5}$$

Example 5: Find the binomial series for (a) $\frac{1}{(1+x)^2}$ (b) $\frac{1}{\sqrt{1+x}}$.

Solution: (a) $1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$,

(b) $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} x^n$.

Theorem Some important Taylor (Maclaurin) series‡.

$$
\begin{aligned}
\frac{1}{1-x} &= \textstyle\sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \cdots, && \text{if} \quad -1 < x < 1 \\
\frac{1}{1+x} &= \textstyle\sum_{n=0}^{\infty} (-1)^n x^n &&= 1 - x + x^2 - x^3 + \cdots, && \text{if} \quad -1 < x < 1 \\
e^x &= \textstyle\sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, && x \in \mathbb{R} \\
e^{-x} &= \textstyle\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} &&= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots, && x \in \mathbb{R} \\
\sin x &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, && x \in \mathbb{R} \\
\cos x &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, && x \in \mathbb{R} \\
\ln(1+x) &= \textstyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} &&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, && \text{if} \quad -1 < x \le 1 \\
\arctan(x) &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} &&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, && \text{if} \quad -1 \le x \le 1
\end{aligned}
$$

Example 6: (a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series. (b) Approximate $\int_0^1 e^{-x^2}\,dx$.

Solution: (a) $\int e^{-x^2}\,dx = \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) dx = C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots$.

(b) $\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots \approx 0.7475$.

‡在此定理，等比級數、逐項微分、逐項積分等技巧可用來求得函數之泰勒級數第 n 項係數 $\frac{1}{n!} f^{(n)}(0)$。