

泰勒級數

114-2 學期 中原大學微積分教學小組¹製

函數 f 在 x_0 的線性近似函數為 $f(x_0) + f'(x_0)(x - x_0)$ 。也就是，我們可用一次多項式 $L(x) = f(x_0) + f'(x_0)(x - x_0)$ 來估計在 x_0 附近的函數值 $f(x)$ 。一般而言，我們可用 n 次多項式更精確地來估計 $f(x)$ ；此多項式稱為泰勒多項式 (*Taylor polynomial*)。

Definition 1 If a function f can be differentiated n times at c , the n th *Taylor polynomial* for f at c is

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n. \quad (1)$$

- Note that $P_n(c) = f(c)$ and $P_n^{(k)}(c) = f^{(k)}(c)$ for $k = 1, 2, \dots, n$.

- When $c = 0$, the Taylor polynomial is also called the *Maclaurin polynomial*; that is, the n th *Maclaurin polynomial* for f is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \quad (2)$$

Example 1: Find the 6th Taylor polynomials at $\frac{\pi}{2}$ for (a) $\sin x$ (b) $\cos x$.

Solution: (a) $P_6(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{24}(x - \frac{\pi}{2})^4 - \frac{1}{720}(x - \frac{\pi}{2})^6$,

(b) $P_6(x) = -(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^3 - \frac{1}{120}(x - \frac{\pi}{2})^5$.

Example 2: Find the n th Taylor polynomial for $1/x$ at 1.

Solution: $P_n(x) = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots + (-1)^n(x - 1)^n$.

Example 3: Find the 7th Maclaurin polynomials for $f(x) =$ (a) $\cos x$ (b) $\frac{1}{1 - x}$ (c) e^x .

Solution: (a) $P_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. (b) $P_7(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$.

(c) $P_7(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$.

Definition 2 If a_0, a_1, a_2, \dots are constants and x is a variable, then a series of the form

$$\sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n + \cdots$$

is called a *power series* (冪級數) in x .

Definition 3 If $f^{(n)}(c)$ exists for all $n = 1, 2, 3, \dots$, the *Taylor series*² (泰勒級數) of $f(x)$ at c is

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots \quad (3)$$

When $c = 0$, the Taylor series is also called the *Maclaurin series*.

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²無窮級數之收斂性檢定法請參閱教科書 (Calculus, 4th edition by Smith and Minton) §9.7~ §9.8.

此類級數屬於 power series 的一種，泰勒多項式 P_n 可視為泰勒級數的前 n 項的和。

冪級數的例子：
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots, \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

Example 4: Express $\frac{1}{1+x^2}$ as a power series if $-1 < x < 1$.

Solution: Using the fact $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots.$$

Binomial Series (二項式級數) If k is a real number and $-1 < x < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots. \quad (4)$$

當 k 為一正整數，二項級數即為二項展開式 (binomial expansion)，即

$$(1+x)^k = \sum_{n=0}^k \binom{k}{n} x^n, \quad \text{其中} \binom{k}{n} = C_n^k = \frac{k!}{n!(k-n)!}, \quad \text{且定義 } 0! = 1. \quad (5)$$

Example 5: Find the binomial series for (a) $\frac{1}{(1+x)^2}$ (b) $\frac{1}{\sqrt{1+x}}$.

Solution: (a) $1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots = \sum_{n=0}^{\infty} (-1)^n (n+1)x^n$,

$$(b) 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} x^n.$$

Theorem Some important Taylor (Maclaurin) series³.

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \dots, && \text{if } -1 < x < 1 \\ \frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n &&= 1 - x + x^2 - x^3 + \dots, && \text{if } -1 < x < 1 \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, && x \in \mathbb{R} \\ e^{-x} &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} &&= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots, && x \in \mathbb{R} \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, && x \in \mathbb{R} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, && x \in \mathbb{R} \\ \ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} &&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, && \text{if } -1 < x \leq 1 \\ \tan^{-1}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} &&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, && \text{if } -1 \leq x \leq 1 \end{aligned}$$

Example 6: (a) Evaluate $\int e^{-x^2} dx$ as an infinite series. (b) Approximate $\int_0^1 e^{-x^2} dx$.

Solution: (a)
$$\int e^{-x^2} dx = \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \right) dx = C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots.$$

$$(b) \int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots \approx 0.7475.$$

³在此定理，等比級數、逐項微分、逐項積分等技巧可用來求得函數之泰勒級數第 n 項係數 $\frac{1}{n!} f^{(n)}(0)$ 。